Exercises :

" An Introduction to Analysis " by William R. Wade

2^{nd} Edition : p. 334 Q10 , p. 339 Q5

Math 2010 D Problem set 5 : Q7

Q10. Let $\mathcal{H}$ be the hyperboloid of one sheet,

given by $x^2+y^2-z^2=1$

a) Prove that at every point $(a,b,c) \in \mathcal{H}$, $\mathcal{H}$ has a tangent plane whose normal is given by $(-a,-b,c)$ $\Rightarrow c=0$

b) Find an equation of each plane tangent to $\mathcal{H}$ which is perpendicular to the xy-plane $(0,0,1)$

c) Find an equation of each plane tangent to $\mathcal{H}$ which is parallel to the plane $x+y-z=1$.

Interchange of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$

does not preserve the equality in general

Q7. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Show that $\frac{\partial f}{\partial y}(x,0) = x$ for all $x \in \mathbb{R}$

and $\frac{\partial f}{\partial x}(0,y) = -y$ for all $y \in \mathbb{R}$.

(b) Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$

(Check Tutorial 7, Math 2010C)

Q5. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable.

Prove that $u(x,y) := f(xy)$ satisfies

$$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 0,$$

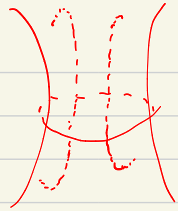
and $v(x,y) := f(x-y) + g(x+y)$ satisfies

the wave equation; i.e.,

$$\frac{\partial^2 v}{\partial^2 x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

Q10 a)

Solution.



$$x^2 + y^2 - z^2 - 1 = 0$$

Let $F(x,y,z) = x^2+y^2-z^2-1$

A normal vector of the tangend plane at (a,b,c) is (by theorem)

$$\nabla F = \left(\frac{\partial F}{\partial x}(a,b,c) , \frac{\partial F}{\partial y}(a,b,c) , \frac{\partial F}{\partial z}(a,b,c) \right)$$

$$= (2a, 2b, -2c) \;//\; (-a,-b,c)$$

b) Perpendicular to xy-plane

(normal vector of xy-plane is $(0,0,1)$)

i.e. $$(-a,-b,c) \cdot (0,0,1) = 0$$

$$c = 0$$

Therefore, the only possible points are

$$\{ (a,b,0) : a^2+b^2 = 1 \}$$

The corresponding tangent plane is

$$-a(x-a) - b \cdot (y-b) + 0 = 0$$

$$-ax - by + a^2 + b^2 = 0$$

$$ax + by = 1$$

c). Parallel to the plane $x+y-z=1$

(normal vector of the plane is $(1,1,-1)$)

i.e. $(-a,-b,c) \parallel (1,1,-1)$

i.e. $a = b = c$

Plugging back to the equation,

$$a^2 + a^2 - a^2 = 1$$

$$a^2 = 1$$

$$a = \pm 1$$

For $a=1$, the tangent plane at $(1,1,1)$ is

$$-1\cdot(x-1) - 1\cdot(y-1) + 1(z-1) = 0$$

$$x+y-z = 1$$

For $a=-1$, the tangent plane at $(-1,-1,-1)$ is

$$1\cdot(x+1) + 1\cdot(y+1) - 1\cdot(z+1) = 0$$

$$x+y-z = -1$$

7(a).

$$\frac{\partial f}{\partial y}(x,0) = \lim_{y\to 0} \frac{f(x,y) - f(x,0)}{y}$$

$$= \begin{cases} \lim\limits_{y\to 0} \dfrac{0}{y} & \text{if } x=0 \\ \lim\limits_{y\to 0} \dfrac{xy\frac{x^2-y^2}{x^2+y^2} - 0}{y} & \text{if } x\neq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x=0 \\ x & \text{if } x\neq 0 \end{cases}$$

$$\frac{\partial f}{\partial x}(0,y) = \lim_{x\to 0} \frac{f(x,y) - f(0,y)}{x}$$

$$= \begin{cases} \lim\limits_{x\to 0} \dfrac{0-0}{x} & \text{if } y=0 \\ \lim\limits_{x\to 0} \dfrac{xy\frac{x^2-y^2}{x^2+y^2} - 0}{x} & \text{if } y\neq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } y=0 \\ -y & \text{if } y\neq 0 \end{cases}$$

(b) $$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x\to 0} \frac{1}{x}\left[\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)\right]$$

$$= 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{1}{y}\left[\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)\right]$$

$$= -1$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$$

5 (i)

$u(x,y) = f(S(x,y))$

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ \cup\!\!\!\shortmid \quad & \qquad \cup\!\!\!\shortmid \\ z &\longmapsto f(z) \end{aligned}$$

We consider

$$\begin{aligned} \hookrightarrow S : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ \cup\!\!\!\shortmid \quad & \qquad \cup\!\!\!\shortmid \\ (x,y) &\longmapsto xy \end{aligned}$$

$$u(x,y) = f(S(x,y))$$

$$\left.\frac{\partial u}{\partial x}\right|_{(a,b)} = \left. f'(z)\right|_{z=S(a,b)} \cdot \left.\frac{\partial S}{\partial x}\right|_{(a,b)}$$

$$= f'(ab) \cdot b$$

$$\left.\frac{\partial u}{\partial y}\right|_{(a,b)} = \left. f'(z)\right|_{z=s(a,b)} \cdot \left.\frac{\partial s}{\partial y}\right|_{(a,b)}$$

$$= f'(ab) \cdot a$$

$$\left.\left[x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right]\right|_{(x,y)=(a,b)}$$

$$= a\,(f'(ab)\cdot b) - b\,(f'(ab)\cdot a)$$

$$= 0$$

(ii) Ex !!

$$\frac{\partial^2 v}{\partial x^2} = f''(x-y) + g''(x+y)$$

$$= \frac{\partial^2 v}{\partial y^2}$$